

# LINEAR MATROID INTERSECTION IS IN QUASI-NC

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**Abstract.** Given two matroids on the same ground set, the matroid intersection problem asks to find a common independent set of maximum size. In case of linear matroids, the problem had a randomized parallel algorithm but no deterministic one. We give an almost complete derandomization of this algorithm, which implies that the linear matroid intersection problem is in quasi-NC. That is, it has uniform circuits of quasi-polynomial size  $n^{O(\log n)}$  and  $O(\text{polylog}(n))$  depth. Moreover, the depth of the circuit can be reduced to  $O(\log^2 n)$  in case of zero characteristic fields. This generalizes a similar result for the bipartite perfect matching problem. Our main technical contribution is to derandomize the Isolation lemma for the family of common bases of two matroids. We use our isolation result to give a quasi-polynomial time black-box algorithm for a special case of *Edmonds' problem*, i.e., *singularity testing of a symbolic matrix*, when the given matrix is of the form  $A_0 + A_1x_1 + \dots + A_mx_m$ , for an arbitrary matrix  $A_0$  and rank-1 matrices  $A_1, A_2, \dots, A_m$ . This can also be viewed as a blackbox polynomial identity testing algorithm for the corresponding determinant polynomial. Another consequence of this result is a deterministic solution to the maximum rank matrix completion problem. Finally, we use our result to find a deterministic representation for the union of linear matroids in quasi-NC.

**Keywords.** Matroid Intersection, Isolation Lemma, Derandomization, Polynomial Identity Testing.

**Subject classification.** 90C57, 68W10, 05B35, 68W20.

## 1. Introduction

Matroids are combinatorial structures that generalize the notion of *linear independence* in Linear Algebra. A matroid  $M$  is a pair  $M = (E, \mathcal{I})$ , where  $E$  is the finite ground set and  $\mathcal{I} \subseteq \mathcal{P}(E)$  is a family of subsets of  $E$  that are said to be the *independent sets*. There are two axioms the independent sets must satisfy: (1) closure under subsets and (2) the *augmentation property* – for any two independent sets of different sizes, the smaller one can be augmented with an element from the bigger one to obtain a new independent set (see the Preliminary Section for exact definitions).

Matroids are motivated by Linear Algebra. For an  $n \times m$  matrix  $V$  over some field, let  $v_1, v_2, \dots, v_m$  be the column vectors of  $V$ , in this order. We define the ground set  $E = \{1, 2, \dots, m\}$  as the set of indices of the columns of  $V$ . A set  $I \subseteq E$  is defined to be independent, if the collection of vectors  $v_i$ , for  $i \in I$ , is linearly independent. Then  $M = (E, \mathcal{I})$  is a matroid: Any subset of an independent set is again independent. The augmentation property is equivalent to the Steinitz Exchange Lemma for two bases of the vector space spanned by the column vectors of  $V$ . A matroid is called *linear*, if it can be represented by a matrix in the above sense over some field.

Although we will formulate most of our results in terms of general matroids, our main result is for *linear* matroids. Hence, for a reader who is unfamiliar with matroid theory, it suffices to think of a matroid simply as a matrix as described above.

The augmentation property implies that all inclusion-wise maximal independent sets have the same size. A maximal independent set is called a *base* of the matroid. The *matroid problem* consists in computing a base of a given matroid. It can be solved efficiently by a simple *greedy algorithm*, provided that we can efficiently test whether a set is independent. There is also a parallel algorithm if we are given a rank oracle for the matroid: for each  $i$ , include the  $i$ -th element in the base if its inclusion to the set of first  $i - 1$  elements increases the rank of the set. See [Karp et al. \(1988\)](#) for parallel complexity of matroid problems under various oracles.

In the *matroid intersection problem*, we are given two matroids  $M_1$  and  $M_2$  over the same ground set. One has to find

the largest set which is independent in both matroids. In the Linear Algebra example, we are given two matrices  $U$  and  $V$  of the same dimensions. We want to compute the largest set  $I$  of indices, such that the columns of  $U$  and the columns of  $V$  indexed by  $I$  are both independent sets. As another example, the bipartite matching problem can be expressed as a matroid intersection problem. An equivalent version of matroid intersection problem asks to find a common base, a set that is a base in both the matroids, if one exists (see [Section 2](#)). A common base generalizes the notion of a perfect matching in a bipartite graph.

The matroid intersection problem can be solved in polynomial time by an algorithm due to [Edmonds \(1968, 1979\)](#). Edmonds' algorithm is a generalization of the famous augmenting path algorithm for bipartite matching. In the case of linear matroids (both represented over the same field), its parallel complexity is also similar to the matching problem. [Narayanan, Saran & Vazirani \(1994\)](#) presented a randomized NC-algorithm based on the *Isolation Lemma*. Applied to matroid intersection, the Isolation Lemma states that randomly chosen weights for the elements of the ground sets isolate a common base, i.e., there is a unique minimum weight common base set, with high probability.

Derandomizing the Isolation Lemma is a major open problem. It would, in particular, imply a *deterministic* parallel algorithm for linear matroid intersection. Recently, the authors together with Fenner ([Fenner et al. 2016](#)) gave an almost complete derandomization of the Isolation Lemma in the case of bipartite perfect matching, and presented a quasi-NC-algorithm for the problem. In the current paper, we generalize the matching algorithm to a quasi-NC-algorithm for linear matroid intersection. Our main result ([Theorem 3.1](#)) is:

*Linear Matroid Intersection is in quasi-NC.*

This puts a rich class of problems in quasi-NC. To give a few examples, besides bipartite perfect matching, the following combinatorial problems NC-reduce to linear matroid intersection (see [Schrijver \(2003\)](#)), and thus, fall into the class quasi-NC.

- Two edge-disjoint spanning trees in a graph
- Rainbow spanning tree in an edge-colored graph
- Arborescence in a directed graph
- Shortest  $R - S$  biconnector and a longest  $R - S$  biforest of a graph

In [Section 4.4](#), we give the definitions of these problems and indicate the reductions to linear matroid intersection. To the best of our knowledge, there is no better bound known on the parallel complexity for these problems, except for the  $r$ -arborescence problem that already had an NC-algorithm due to [Lovász \(1985\)](#). The second and the last problem generalize the bipartite matching problem, for which quasi-NC is still the best known bound.

Our main technique is to deterministically construct a weight assignment that isolates a common base of the two given matroids. Hence, this can again be seen as a derandomization of the Isolation Lemma in this setting. Following the approach of the bipartite matching result ([Fenner et al. 2016](#)), we look at the isolation question in the corresponding polytope. However, since the matroid intersection polytope has a more complicated description than the bipartite matching polytope, we need more ideas. The novel part is to analyze the faces of the matroid intersection polytope ([Section 3.2](#)) and to come up with an appropriate definition of *cycles in the intersection of two matroids* ([Section 3.3](#)). As before, our weights have  $O(\log^2 n)$  bits, and so we obtain circuits of quasi-polynomial size  $n^{O(\log n)}$ . Hence, we get linear matroid intersection in quasi-NC.

It remains open whether the problem is in NC. We would like to point out that our isolating weight assignment actually works for general matroid intersection and even for polymatroid intersection. However, we get the quasi-NC-bound only in the case of linear matroids, because only there do we have a connection to the determinant.

Subsequent to this work, our derandomization of the Isolation Lemma has been generalized to a larger class of families of sets ([Gurjar et al. 2017](#)).

**1.1. Polynomial identity testing (PIT) and singularity of symbolic matrices.** Our derandomization of the Isolation Lemma in the above setting also gives a blackbox polynomial identity testing algorithm for an interesting class of polynomials. The *polynomial identity problem* asks whether a given multivariate polynomial is the zero-polynomial. The polynomial can be given, for example, as an arithmetic circuit, an arithmetic branching program, or a symbolic matrix. In the latter case, the polynomial is defined as the determinant of the symbolic matrix. Given a polynomial in one of these representations, it might take exponential time to compute an explicit representation as a sum of monomials. However, evaluating the polynomial at a point is easy, and this suffices for an easy randomized polynomial identity test: just evaluate the polynomial at a random point. It is known that a nonzero polynomial will have a nonzero evaluation with high probability (Demillo & Lipton 1978; Schwartz 1980; Zippel 1979). However, no nontrivial deterministic tests are known. Deterministic PIT is known to have connections with arithmetic circuit lower bounds (Agrawal 2005; Kabanets & Impagliazzo 2003).

The *singularity problem of symbolic matrices*, also known as *Edmonds' problem* (Edmonds 1967), is a special case of PIT. Given a matrix  $A$  whose entries are linear forms in a set of formal variables, one has to determine whether  $A$  is singular, i.e., whether  $\det(A)$  is the zero-polynomial. This problem captures PIT for small degree arithmetic circuits, with only a quasi-polynomial blow-up (Valiant 1979; Valiant *et al.* 1983). Efficient polynomial singularity tests are known only for very restricted cases. One such case which has received a lot of attention is when  $A$  is of the form  $A = \sum_i z_i A_i$ , where the  $A_i$ 's are rank-1 matrices (Edmonds 1967; Lovász 1989; Murota 1993). Singularity testing for this case corresponds exactly to the linear matroid intersection question (see Section 2.4 and Section 4.1), and thus has a polynomial-time algorithm (Edmonds 1979; Lovász 1989). However, no non-trivial *blackbox* PIT algorithm was known for this case. A blackbox algorithm is one which is not given the description of the input polynomial, but only the size parameters. In our case, the algorithm does not use the entries of the given matrices, just the number of matrices and

their dimension. The goal is to construct a *hitting set*, a set of points such that if the polynomial is nonzero, then it evaluates to nonzero at least at one of the points. In case of finite fields, a hitting set is allowed to have points from a sufficiently large field extension. With our derandomization of the Isolation Lemma we get a construction of a hitting set for  $\det(\sum_i z_i A_i)$ , when the  $A_i$ 's are of rank 1.

As a generalization of the above, we add an arbitrary constant matrix  $A_0$ , i.e., we consider matrices of the form  $A = A_0 + \sum_i z_i A_i$ . There is a polynomial-time whitebox algorithm to determine the singularity of such  $A$ 's (Geelen 1999; Ivanyos *et al.* 2010; Murota 1993). Using reductions from Anderson, Shpilka & Volk (2016) and Murota (1993), our hitting set from above also works for this case (Theorem 4.2).

In quasi-polynomial time, we can construct a hitting set for polynomials of the form  $\det(A_0 + \sum_{i=1}^m z_i A_i)$ , where  $A_0$  is an arbitrary matrix and  $A_i$  is a matrix of rank 1, for  $1 \leq i \leq m$ .

This result can also be used for a blackbox solution to another version of *Edmonds' problem* (Edmonds 1967): given a set of matrices, find a matrix of maximum rank in the linear span of these matrices. In the case when the given matrices are of rank 1, one of the points in our hitting set provides the linear combination of the given matrices which achieves maximum rank. To see this, suppose the given matrices are  $A_1, A_2, \dots, A_m$  and consider the symbolic matrix  $A = \sum_i z_i A_i$ . Let  $\sum_i \alpha_i A_i$  be a linear combination that achieves the maximum rank, say  $r$ . Then, there are  $r \times r$  sub-matrices  $B_i$  of  $A_i$ , for  $i = 1, \dots, m$  such that  $\sum_i \alpha_i B_i$  is non-singular. Thus,  $B = \sum_i z_i B_i$  is non-singular as well. Note that matrices  $B_i$  have rank 1. Therefore, our hitting set for  $\det(A)$  also works for  $\det(B)$ . That is, when the variables  $\mathbf{z} = (z_1, z_2, \dots, z_m)$  are substituted with points in our hitting set,  $\det(B(\mathbf{z} = \alpha))$  will be nonzero for at least one of the points  $\alpha$ . For such a point  $\alpha$ , matrix  $A(\mathbf{z} = \alpha)$  has the maximum rank. That is,  $\alpha$  is the desired linear combination that solves Edmonds' problem.

**1.2. Maximum rank matrix completion.** The above PIT result also provides a blackbox solution to the *maximum rank matrix completion problem*. Given a partially filled matrix, the objective is to fill in the blank entries so as to maximize the rank of the matrix. Again, in case of finite fields, the entries are allowed to be from a large enough field extension. There is a simple randomized solution: filling in random values for the blank entries achieves the maximum rank with high probability (Demillo & Lipton 1978; Schwartz 1980; Zippel 1979). The argument goes via a reduction to PIT (see Section 4.2). The problem also had a deterministic polynomial time algorithm (see Geelen (1999); Ivanyos *et al.* (2010); Murota (1993)). What is currently open is the question of finding a polynomial time blackbox solution, that is, filling in the blank entries without looking at the given matrix. We do this in quasi-polynomial time.

Given a partially filled matrix, in quasi-polynomial time, we can compute a fixed substitution for the blank entries (with quasi-polynomially many bits) which maximizes the rank of the matrix for all choices of the already filled in entries.

**1.3. A representation for matroid union.** For matroids  $M_1, M_2, \dots, M_k$  with ground sets  $E_1, E_2, \dots, E_k$ , respectively, the *matroid union*  $M = M_1 \vee M_2 \vee \dots \vee M_k$  is a matroid with ground set  $\bigcup_{i=1}^k E_i$ . A set  $I$  is independent in  $M$ , if  $I = \bigcup_{i=1}^k I_i$ , for independent sets  $I_1, I_2, \dots, I_k$  of  $M_1, M_2, \dots, M_k$ , respectively. Note that the sets  $E_1, E_2, \dots, E_k$  need not be disjoint, and as we will see later, the main difficulty lies when there is an overlap between these sets. The problem of finding a maximum independent set in a matroid union reduces to matroid intersection (see Section 4.3). Thus, this problem is in quasi-NC for linear matroids.

When  $M_1, M_2, \dots, M_k$  are linear matroids represented over the same field, then their union  $M$  is also linear. An interesting question is to find a matrix that represents  $M$ . Narayanan, Saran & Vazirani (1994) present a randomized parallel algorithm for this problem. Our PIT result from above gives a quasi-polynomial time solution for this question (Theorem 4.9).

Given matrices representing the linear matroids  $M_1, M_2, \dots, M_k$  over the same field, we can deterministically construct a matrix representing the matroid union  $M_1 \vee M_2 \vee \dots \vee M_k$  whose entries have a quasi-polynomially bounded size.

## 2. Preliminaries

For a set  $E$ , we denote the power set of  $E$  by  $\mathcal{P}(E)$ . For an integer  $m$ , we define  $[m] = \{1, 2, \dots, m\}$ .

**2.1. Complexity classes.** Barrington (1992) generalized the class  $\text{NC}^k$  to define a class  $\text{quasi-NC}^k$  as the class of problems which have uniform circuits of quasi-polynomial size  $2^{\log^{O(1)} n}$  and polylogarithmic depth  $O(\log^k n)$ . The class  $\text{quasi-NC}$  is the union of classes  $\text{quasi-NC}^k$ , over all  $k \geq 0$ . The notion of *uniformity* that Barrington (1992) used was deterministic poly-log time uniformity, i.e., given two gate numbers in the circuit, it should be possible to decide the parent-child relation in poly-log time. Here, we use a more relaxed notion of *uniformity* which requires that the circuit should be computable in polylogarithmic space. The reason is that we use determinant computation as a major subroutine and to the best of our knowledge, the existing results showing that determinant is in  $\text{NC}$  do it with the notion of logspace uniformity.

**2.2. Matroids.** Matroid theory originated in the middle of the 1930s. There is a huge literature on matroids by now. For an introduction, see for example the excellent textbooks of Oxley (2006) or Schrijver (2003). Below we give some basic definitions and facts about matroids.

A *matroid*  $M$  is a pair  $M = (E, \mathcal{I})$ , where  $E$  is the finite ground set and  $\mathcal{I} \subseteq \mathcal{P}(E)$  is a nonempty family of subsets of  $E$  that satisfies the following two axioms.

1. *Closure under subsets.* For every  $I \in \mathcal{I}$  and  $J \subseteq I$  we have  $J \in \mathcal{I}$ .
2. *Augmentation property.* For every  $I, J \in \mathcal{I}$  where  $|I| < |J|$ , there is a  $j \in J$  such that  $I \cup \{j\} \in \mathcal{I}$ .



We denote  $m = |E|$  throughout the paper. The sets in  $\mathcal{I}$  are called the *independent sets of  $M$* . An inclusion-wise maximal set  $B \in \mathcal{I}$  is called a *base*. Note that by the augmentation property, all base sets have the same size. Let  $\mathcal{B} \subseteq \mathcal{I}$  denote the collection of base sets.

As an example, we already mentioned *linear matroids* in the Introduction which come from linear independence in Linear Algebra. A very simple subclass of linear matroids are *partition matroids*. Such a matroid is given by a partition  $B_1, B_2, \dots, B_k$  of the ground set  $E$ , and numbers  $b_1, b_2, \dots, b_k$ . A set  $I \subseteq E$  is independent, if  $|I \cap B_i| \leq b_i$ , for all  $i = 1, 2, \dots, k$ .

Another well-known example are *graphic matroids*. Given an undirected graph  $G = (V, E)$ , we take  $E$  as the ground set and the forests in  $G$  as the independent sets. It is not hard to see that forests fulfill the matroid axioms.

**Matroid rank.** Motivated by Linear Algebra, there is a *rank-function* of a matroid that is defined for every subset  $A \subseteq E$  as the size of the largest independent set that is contained in  $A$ ,

$$\text{rank}(A) = \max\{|I| \mid I \in \mathcal{I} \text{ and } I \subseteq A\}.$$

The size of every maximal independent set is  $\text{rank}(E)$ . This number is called the *rank of  $M$* .

An important property of the rank-function is its submodularity. In general, a function  $f: \mathcal{P}(E) \rightarrow \mathbb{R}$  is called *submodular*, if for any sets  $S, T \subseteq E$ , we have

$$f(S) + f(T) \geq f(S \cup T) + f(S \cap T).$$

LEMMA 2.1 (See [Schrijver 2003](#)). *The rank-function of a matroid is submodular.*

PROOF. Let  $S, T \subseteq E$ . Let  $I, J \in \mathcal{I}$  be maximal such that  $I \subseteq S \cap T$  and  $I \subseteq J \subseteq S \cup T$ . Hence,  $\text{rank}(S \cap T) = |I|$  and  $\text{rank}(S \cup T) = |J|$ .

Define  $S' = J \cap S$  and  $T' = J \cap T$ . Note that  $S', T' \in \mathcal{I}$  and  $S' \cap T' = I$ . Hence, we get

$$\begin{aligned} r(S) + r(T) &\geq |S'| + |T'| = |S' \cup T'| + |S' \cap T'| \\ &\geq |J| + |I| \\ &= r(S \cup T) + r(S \cap T). \end{aligned}$$

This proves the lemma.  $\square$

**Dual Matroid.** There is a concept of *duality* in matroid theory. Let  $M = (E, \mathcal{I})$  be a matroid with base sets  $\mathcal{B}$ . Define  $\mathcal{B}^*$  as the complements of the base sets,  $\mathcal{B}^* = \{\overline{B} \mid B \in \mathcal{B}\}$ . Then  $\mathcal{B}^*$  are the base sets of a matroid  $M^*$ , the *dual* of  $M$ . In terms of independent sets, we can write  $M^* = (E, \mathcal{I}^*)$ , where

$$\mathcal{I}^* = \{I \mid \exists B \in \mathcal{B} \ B \cap I = \emptyset\}.$$

It is known that the dual of a linear matroid is again linear. Moreover, given the matrix that represents a linear matroid, the matrix that represents the dual matroid can be computed in  $\text{NC}^2$  (Narayanan *et al.* 1994).

**Matroid intersection.** Our main focus is the *matroid intersection problem*. Given two matroids  $M_1 = (E, \mathcal{I}_1)$  and  $M_2 = (E, \mathcal{I}_2)$  over the same ground set, compute a maximum size set in  $\mathcal{I}_1 \cap \mathcal{I}_2$ , the common independent sets. Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be the collections of base sets of  $M_1$  and  $M_2$ , respectively. In another variant of the problem, one has to decide whether the matroids have a common base, i.e., whether  $\mathcal{B}_1 \cap \mathcal{B}_2$  is nonempty, and in this case, to construct such a base  $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ . The two variants are equivalent for linear matroids. The reduction from maximum common independent set to common base is implicit in Narayanan *et al.* (1994, Theorem 4.2). Note that in general,  $(E, \mathcal{I}_1 \cap \mathcal{I}_2)$  is not a matroid anymore. We should clarify that in the linear matroid intersection problem, it is usually assumed that both the matroids are linearly represented over the *same field*.

Matroid intersection captures many interesting combinatorial problems.

- We already mentioned the problem of common linear independent columns of two matrices (over the same field).
- A well-known example is *bipartite maximum matching*. Let  $G = (L \cup R, E)$  be a bipartite graph. We define two partition matroids  $M_L$  and  $M_R$  over the ground set  $E$ . In matroid  $M_L$ , for  $v \in L$ , define sets  $B_v = \{e \in E \mid v \in e\}$  that partition  $E$ . A set  $I \subseteq E$  is independent, if  $|I \cap B_v| \leq 1$ , for all  $v \in L$ , i.e., if no two edges have a common end point in  $L$ . Matroid  $M_R$  is defined similarly with respect to vertex set  $R$ . Then, any common independent set of  $M_L$  and  $M_R$  is a matching in the graph  $G$ . Note that  $M_L$  and  $M_R$  are linear matroids.

We provide more examples in [Section 4.4](#).

**2.3. Matroid polytope.** The polytopes we consider in this paper are *convex polytopes* defined as the convex hull of finitely many points in  $\mathbb{R}^m$ . Any convex polytope  $P$  can be described as the intersection of halfspaces, i.e., as  $P = \{x \in \mathbb{R}^m \mid Ax \leq b\}$ , for some matrix  $A \in \mathbb{R}^{k \times m}$  and vector  $b \in \mathbb{R}^k$ . A *face*  $F$  of the polytope  $P$  is a set of points in  $P$  minimizing or maximizing a linear function. For example, for some vector  $w \in \mathbb{R}^m$ , we have

$$F = \{x \in P \mid w \cdot x = \min_{y \in P} w \cdot y\}.$$

If the polytope  $P$  is described by  $Ax \leq b$ , then any face of  $P$  can be described as  $\{x \in P \mid A'x = b'\}$ , where  $(A' \ b')$  is some subset of the rows of  $(A \ b)$ .

With every matroid, there is an associated *matroid polytope*. This polytope is crucial for our arguments.

For a set  $S \subseteq E$ , its *characteristic vector*  $\chi^S \in \mathbb{R}^E$  is defined as

$$\chi_e^S = \begin{cases} 1, & \text{if } e \in S, \\ 0, & \text{otherwise.} \end{cases}$$

For any collection of sets  $\mathcal{A} \subseteq \mathcal{P}(E)$ , the polytope  $P(\mathcal{A}) \subset \mathbb{R}^E$  is defined as the convex hull of the characteristic vectors of the sets in  $\mathcal{A}$ ,

$$P(\mathcal{A}) = \text{conv}\{\chi^I \mid I \in \mathcal{A}\}.$$

For a matroid  $M = (E, \mathcal{I})$ , its *matroid polytope* is defined as  $P(\mathcal{I}) \subset \mathbb{R}^E$ , i.e., the convex hull of the characteristic vectors of the independent sets. The points  $\{\chi^I \mid I \in \mathcal{I}\}$  are the corners of the matroid polytope  $P(\mathcal{I})$ .

For a family  $\mathcal{B}$  of base sets of the matroid  $(E, \mathcal{I})$ , the *matroid base polytope* is defined as  $P(\mathcal{B})$ . It is a face of the matroid polytope  $P(\mathcal{I})$  because the set of points in  $P(\mathcal{I})$  which maximize the function  $(1, 1, \dots, 1) \cdot x$  is exactly  $P(\mathcal{B})$ .

Edmonds (1970) gave a simple description of this polytope which uses the rank function of the matroid (see also Schrijver (2003)). For convenience, we define for any  $x \in \mathbb{R}^E$  and  $S \subseteq E$ ,

$$x(S) = \sum_{e \in S} x_e.$$

Note that  $x(S)$  can be written as an inner product,  $x(S) = x \cdot \chi^S$ .

LEMMA 2.2 (Edmonds 1970). *Let  $(E, \mathcal{I})$  be a matroid of rank  $n$  with rank function  $r$ , and  $x \in \mathbb{R}^E$ . Then,  $x \in P(\mathcal{I})$  iff*

$$(2.3) \quad x_e \geq 0 \quad \forall e \in E$$

$$(2.4) \quad x(S) \leq r(S) \quad \forall S \subseteq E.$$

Similarly, for the matroid base polytope,  $x \in P(\mathcal{B})$  if and only if (2.3) and (2.4) hold, and

$$(2.5) \quad x(E) = n.$$

It is easy to see that any 0-1 corner of the polytope given by (2.3) and (2.4) corresponds to an independent set in  $\mathcal{I}$ . The nontrivial part is to show that the described polytope does not have a non-integral corner.

**Matroid intersection polytope.** The intersection of two matroids also has an easy polytope description: Edmonds (1970) showed a surprising result that one can describe the matroid intersection polytope  $P(\mathcal{I}_1 \cap \mathcal{I}_2)$  just by putting together the constraints of the two matroid polytopes  $P(\mathcal{I}_1)$  and  $P(\mathcal{I}_2)$  (see also Schrijver (2003)). The same holds for the matroid base polytope.

**THEOREM 2.6** (Edmonds 1970). *Let  $(E, \mathcal{I}_1)$  and  $(E, \mathcal{I}_2)$  be two matroids with base sets  $\mathcal{B}_1, \mathcal{B}_2$  and rank functions  $r_1, r_2$ , respectively. Then*

$$P(\mathcal{I}_1 \cap \mathcal{I}_2) = P(\mathcal{I}_1) \cap P(\mathcal{I}_2).$$

*That is, for a point  $x \in \mathbb{R}^E$ , we have  $x \in P(\mathcal{I}_1 \cap \mathcal{I}_2)$  iff*

$$(2.7) \quad x_e \geq 0 \quad \forall e \in E,$$

$$(2.8) \quad x(S) \leq r_1(S) \quad \forall S \subseteq E,$$

$$(2.9) \quad x(S) \leq r_2(S) \quad \forall S \subseteq E.$$

*Similarly, for the matroid base polytope,  $P(\mathcal{B}_1 \cap \mathcal{B}_2) = P(\mathcal{B}_1) \cap P(\mathcal{B}_2)$ . That is,  $x \in P(\mathcal{B}_1 \cap \mathcal{B}_2)$  iff (2.5), (2.7), (2.8) and (2.9) hold.*

#### 2.4. An RNC-algorithm for linear matroid intersection.

Narayanan, Saran & Vazirani (1994) showed that the linear matroid intersection problem is in RNC. Their technique was to reduce the problem to a polynomial identity test (PIT), namely whether the determinant of a symbolic matrix is nonzero. Moreover, they show that to construct a maximum common independent set, it suffices to have an isolating weight function for a family of common bases of two matroids.

Let  $w: E \rightarrow \mathbb{Z}$  be a weight function. The *weight of a set  $B \subseteq E$*  is defined as  $w(B) = \sum_{e \in B} w(e)$ .

**DEFINITION 2.10.** *A weight function  $w: E \rightarrow \mathbb{Z}$  is isolating for a family of sets  $\mathcal{A} \subseteq \mathcal{P}(E)$ , if there is a unique minimum weight set in  $\mathcal{A}$ .*

We give some details on the argument, because we will use the same algorithm, except that we will *deterministically* compute the isolating weight function.

Let the linear matroids  $M_1$  and  $M_2$  be given by two matrices  $U$  and  $V$ . We want to find out whether  $M_1$  and  $M_2$  have a common base. Note that to have a common base, the two matrices must have the same rank. Without loss of generality, we can assume that both matrices are  $n \times m$  and have full row rank. This is because if the row rank is not full, then one can just keep a basis of the rows and this does not change the matroid.

LEMMA 2.11. *Let  $Z$  be an  $m \times m$  diagonal matrix with variables on the diagonal,  $Z_{e,e} = z_e$ , for  $e = 1, 2, \dots, m$ . Define the  $n \times n$  symbolic matrix  $D = UZV^\top$ . Then  $M_1$  and  $M_2$  have a common base  $\iff \det(D) \neq 0$ .*

PROOF. By the Binet-Cauchy formula, we can write

$$\det(D) = \sum_{\substack{B \subseteq [m] \\ |B|=n}} \left( \prod_{e \in B} z_e \right) \det(U_B) \det(V_B),$$

where  $U_B$  and  $V_B$  are submatrices of  $U$  and  $V$ , respectively, with columns indexed by  $B$ . Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be the collections of bases for  $M_1$  and  $M_2$ , respectively. Clearly,  $\det(U_B) \det(V_B) \neq 0$  if and only if  $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ . Hence, the monomials of  $\det(D)$  are coming precisely from the common bases,

$$(2.12) \quad \det(D) = \sum_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \left( \prod_{e \in B} z_e \right) \det(U_B) \det(V_B).$$

This proves the lemma.  $\square$

Now, let  $w$  be an isolating weight assignment for  $\mathcal{B}_1 \cap \mathcal{B}_2$ . Replace each variable  $z_e$  in equation (2.12) by  $z^{w(e)}$ , for a new variable  $z$ . Then  $\det(D)$  becomes a univariate polynomial  $\det(D)(z)$ . And the monomial  $\prod_{e \in B} z_e$  in equation (2.12) becomes  $z^{w(B)}$  in  $\det(D)(z)$ .

If  $\mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$ , then the minimum degree term in  $\det(D)(z)$  is unique, as  $w$  is isolating. Thus,  $\det(D)(z) \neq 0 \iff \mathcal{B}_1 \cap \mathcal{B}_2 \neq \emptyset$ .

The RNC-algorithm now simply uses random weights. The Isolation Lemma (Mulmuley *et al.* 1987) states that a random weight function  $w$  with polynomially bounded weights is isolating for any family  $\mathcal{A}$  with high probability. Moreover, the determinant polynomial  $\det(D)(z)$  can be computed in NC, when the entries are small degree univariate polynomials (Borodin *et al.* 1984).

THEOREM 2.13 (Narayanan *et al.* 1994). *Linear Matroid Intersection is in RNC.*

One can also compute the common base set  $B^*$  that is isolated. For each  $e \in E$ , in parallel, delete  $e$  and re-compute  $\det(D)(z)$ . If the minimum term disappears, then  $e \in B^*$ .

### 3. Linear matroid intersection in quasi-NC

In this section, we show how to derandomize the algorithm from [Theorem 2.13](#).

**THEOREM 3.1.** *Linear Matroid Intersection is in quasi-NC.*

**REMARK 3.2.** *The precise quasi-NC-level in [Theorem 3.1](#) depends on the underlying field. In general, we get quasi-NC<sup>4</sup>, but for fields of characteristic zero, we can get it in quasi-NC<sup>2</sup>.*

In the RNC-algorithm described in [Section 2.4](#), random weights were used to isolate a base in the intersection of two matroids. We will construct an isolating weight assignment *deterministically*.

We build the isolating weight assignment in rounds. In every round, we slightly modify the current weight assignment to get a smaller set of *minimum* weight common bases. Our goal is to reduce their number in every round significantly. We stop when we have a unique minimum weight common base.

To get a picture of the set of minimum weight common bases with respect to a weight assignment  $w$ , we view  $w$  as a function on the common base polytope. That is, we define an extension of weight function  $w : E \rightarrow \mathbb{Z}$  to  $\mathbb{R}^E$ . For  $x \in \mathbb{R}^E$ ,

$$w(x) = w \cdot x = \sum_{e \in E} w(e) x_e.$$

Note that  $w(\chi^S) = w(S)$ , for any set  $S \subseteq E$ . Now, consider the points minimizing the function  $w(x)$  in the common base polytope. As  $w(x)$  is linear, these points will form a face of the polytope. There will be a one to one correspondence between the corners of this face and the minimum weight common bases. Therefore, we want to understand the properties of such faces. We start by

considering the faces of a base polytope for a single matroid in [Section 3.1](#), and then consider the intersection of two matroids in [Section 3.2](#). The common base polytope and its faces will only be a part of the argument and not of the actual weight construction algorithm.

**3.1. Faces of the matroid polytope.** Let  $(E, \mathcal{I})$  be a matroid with the family of base sets  $\mathcal{B}$  and rank function  $r$ . From the description of the polytope  $P(\mathcal{B})$  in [Lemma 2.2](#), we know that any of its faces can be described by equations of the type  $x_e = 0$  or  $x(S) = r(S)$ . The submodularity of the rank function ensures that the collection of sets  $S$  for which the second equation holds is closed under union and intersection.

**LEMMA 3.3** ([Edmonds 1970](#)). *For any point  $x \in P(\mathcal{B})$  and any sets  $S, T \subseteq E$ , if  $x(S) = r(S)$  and  $x(T) = r(T)$  then*

$$x(S \cap T) = r(S \cap T) \text{ and } x(S \cup T) = r(S \cup T).$$

**PROOF.** From the lemma hypothesis,

$$\begin{aligned} r(S) + r(T) &= x(S) + x(T) = x(S \cup T) + x(S \cap T) \\ &\leq r(S \cup T) + r(S \cap T) \\ &\leq r(S) + r(T). \end{aligned}$$

The first inequality is true since  $x$  satisfies [\(2.4\)](#). The second inequality is true by submodularity ([Lemma 2.1](#)). Thus, all the inequalities are in fact equalities. Hence, the claim follows.  $\square$

For any face  $F$  of the base polytope, [Lemma 3.3](#) gives us a structured basis for all the rank constraints that are tight for  $F$ , i.e., where  $x(T) = r(T)$ , for all  $x \in F$ . The basis consists of equations corresponding to sets that form a partition of the ground set  $E$ .



LEMMA 3.4. *Let  $(E, \mathcal{T})$  be a matroid with family of base sets  $\mathcal{B}$  and rank function  $r$ . Let  $F$  be a face of the matroid base polytope  $P(\mathcal{B})$ . Then, there exists a family  $\mathcal{S}$  of disjoint sets that form a partition of  $E$ , such that for any  $S \in \mathcal{S}$  there exists a number  $\nu_S \geq 0$  such that for any  $x \in F$ ,*

$$x(S) = \nu_S.$$

Moreover,

- (i) *if for some  $T \subseteq E$ ,  $x(T) = r(T)$  for all  $x \in F$ , then  $T$  is a union of sets from  $\mathcal{S}$ ,*
- (ii) *if for some  $e \in E$ ,  $x_e = 0$  for all  $x \in F$ , then there is an  $S \in \mathcal{S}$  such that  $S = \{e\}$  and  $\nu_S = 0$ .*

PROOF. We consider the equations of type  $x(T) = r(T)$  in  $F$ ,

$$\mathcal{T} = \{T \subseteq E \mid x(T) = r(T) \quad \forall x \in F\}.$$

Let  $\mathcal{T} = \{T_1, T_2, \dots, T_p\}$ . Consider the family of sets

$$\mathcal{S} = \{R_1 \cap R_2 \cap \dots \cap R_p \mid R_i \in \{T_i, \overline{T}_i\} \text{ for } i = 1, 2, \dots, p\}.$$

Clearly, the sets in  $\mathcal{S}$  form a partition of  $E$ . We will show that for any  $S \in \mathcal{S}$ , there exists a number  $\nu_S$  such that  $x(S) = \nu_S$ , for all  $x \in F$ .

W.l.o.g. let  $S = T_1 \cap \dots \cap T_j \cap \overline{T}_{j+1} \cap \dots \cap \overline{T}_p$ . Let us denote  $S' = T_1 \cap \dots \cap T_j$  (for  $j = 0$ , let  $S' = E$ ), and  $S'' = T_{j+1} \cup \dots \cup T_p$  (for  $j = p$ , let  $S'' = \emptyset$ ). Then, we have  $S = S' - (S' \cap S'')$ . As  $x(T_i) = r(T_i)$ , for each  $1 \leq i \leq p$ , we get from [Lemma 3.3](#)

$$x(S') = r(S') \quad \text{and} \quad x(S'') = r(S'').$$

Again by [Lemma 3.3](#), we have  $x(S' \cap S'') = r(S' \cap S'')$ . Now,

$$x(S) = x(S') - x(S' \cap S'') = r(S') - r(S' \cap S'').$$

Hence, for  $\nu_S = r(S') - r(S' \cap S'')$ , we have  $x(S) = \nu_S$ .

Claim (i) follows directly from the definition of  $\mathcal{S}$ . For claim (ii), consider an element  $e \in E$  such that  $x_e = 0$  for all  $x \in F$ . For any  $x \in F$ , we have  $x(E - \{e\}) = x(E) - x_e = n = r(E - \{e\})$ . Thus,  $E - \{e\} \in \mathcal{T}$ . We claim that  $\{e\} \in \mathcal{S}$ . To see this, define  $R_i$  to be  $T_i$  or  $\overline{T}_i$ , whichever contains  $e$ . Then clearly,  $R_1 \cap R_2 \cap \dots \cap R_p = \{e\}$ . □

**3.2. Faces of the matroid intersection polytope.** Suppose  $(E, \mathcal{I}_1)$  and  $(E, \mathcal{I}_2)$  are two matroids with family of base sets  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and rank functions  $r_1$  and  $r_2$ , respectively. By [Theorem 2.6](#), the faces of polytope  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  can be described by replacing some of the inequalities [\(2.7\)](#), [\(2.8\)](#), and [\(2.9\)](#) by equalities. This basically means that any face  $F$  of  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  can be written as  $F = F_1 \cap F_2$ , for some faces  $F_1, F_2$  of  $P(\mathcal{B}_1)$  and  $P(\mathcal{B}_2)$ , respectively. Using this fact, we get the following extension of [Lemma 3.4](#) that will be crucial for our weight assignment design.

**LEMMA 3.5.** *Let  $(E, \mathcal{I}_1)$  and  $(E, \mathcal{I}_2)$  be two matroids with families of base sets  $\mathcal{B}_1$  and  $\mathcal{B}_2$  and rank functions  $r_1$  and  $r_2$ , respectively. Let  $F$  be a face of the matroid intersection base polytope  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . Then, there exist two families of disjoint sets  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , each forming a partition of  $E$ , such that for any  $S_1 \in \mathcal{S}_1$  and  $S_2 \in \mathcal{S}_2$  there exist numbers  $\nu_{S_1}, \mu_{S_2} \geq 0$  such that for any  $x \in F$ ,*

$$x(S_1) = \nu_{S_1} \quad \text{and} \quad x(S_2) = \mu_{S_2}.$$

Moreover,

- (i) if for some  $T \subseteq E$ ,  $x(T) = r_1(T)$  for all  $x \in F$ , or  $x(T) = r_2(T)$  for all  $x \in F$ , then  $T$  is a union of sets from  $\mathcal{S}_1$ , respectively  $\mathcal{S}_2$ ,
- (ii) if for some  $e \in E$ ,  $x_e = 0$  for all  $x \in F$ , then there is a  $S_1 \in \mathcal{S}_1$  and a  $S_2 \in \mathcal{S}_2$  such that  $S_1 = S_2 = \{e\}$  and  $\nu_{S_1} = \mu_{S_2} = 0$ .

**PROOF.** We define sets for each type of equality of face  $F$ ,

$$\begin{aligned} S_0 &= \{e \in E \mid x_e = 0 \quad \forall x \in F\}, \\ \mathcal{T}_1 &= \{T \subseteq E \mid x(T) = r_1(T) \quad \forall x \in F\}, \\ \mathcal{T}_2 &= \{T \subseteq E \mid x(T) = r_2(T) \quad \forall x \in F\}. \end{aligned}$$

Now, define faces  $F_1$  and  $F_2$  of polytopes  $P(\mathcal{B}_1)$  and  $P(\mathcal{B}_2)$ , respectively, as

$$\begin{aligned} F_1 &= \{x \in P(\mathcal{B}_1) \mid x(S_0) = 0 \text{ and } x(T) = r_1(T) \quad \forall T \in \mathcal{T}_1\}, \\ F_2 &= \{x \in P(\mathcal{B}_2) \mid x(S_0) = 0 \text{ and } x(T) = r_2(T) \quad \forall T \in \mathcal{T}_2\}. \end{aligned}$$

By [Theorem 2.6](#), we have  $F = F_1 \cap F_2$ . Applying [Lemma 3.4](#) to  $F_1$  and  $F_2$  proves the lemma.  $\square$

**3.3. Cycles in matroid intersection.** Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be the base sets of matroids  $(E, \mathcal{I}_1)$  and  $(E, \mathcal{I}_2)$ , respectively. As mentioned earlier, we will construct the weight assignment in rounds. In each round, we want the dimension of the face of minimum weight common bases to become smaller. To measure this decrement, we define a *cycle* with respect to a face.

**DEFINITION 3.6 (Cycle).** *Let  $F$  be a face of the polytope  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  with the partitions  $\mathcal{S}_1$  and  $\mathcal{S}_2$  as in [Lemma 3.5](#). A sequence  $C = (e_1, e_2, \dots, e_{2r})$  of distinct elements of  $E$  is called a cycle with respect to face  $F$ , if consecutive pairs are alternately in a set from  $\mathcal{S}_1$  and a set from  $\mathcal{S}_2$ . That is, for  $i = 1, 2, \dots, r$ ,*

$$\begin{aligned} e_{2i-1}, e_{2i} &\in S_{2i-1}, & \text{for some } S_{2i-1} \in \mathcal{S}_1, \\ e_{2i}, e_{2i+1} &\in S_{2i}, & \text{for some } S_{2i} \in \mathcal{S}_2, \end{aligned}$$

where  $e_{2r+1} = e_1$ .

To motivate the definition, note that when we view bipartite matching as matroid intersection, then the cycles defined here are exactly the cycles in the corresponding graph.

Note that if every point in face  $F$  satisfies equation  $x_e = 0$  for some element  $e \in E$ , then  $e$  cannot appear in any cycle defined with respect to  $F$ . This is because  $\{e\}$  appears as a singleton set in both the partitions constructed for  $F$ .

First we show that cycles always exist as long as there are at least two bases in the face. This is analogous to the property that in any graph, the symmetric difference of two perfect matchings is a union of disjoint cycles in the graph.

**LEMMA 3.7.** *Let  $B_1, B_2$  be two bases in the face  $F$  of polytope  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . Then  $B_1 \triangle B_2$  is a union of disjoint cycles.*

**PROOF.** Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be the two partitions of  $E$  as in [Lemma 3.5](#). Then, we have

$$(3.8) \quad |B_1 \cap S_1| = |B_2 \cap S_1| = \nu_{S_1}, \quad \text{for every } S_1 \in \mathcal{S}_1$$

$$(3.9) \quad |B_1 \cap S_2| = |B_2 \cap S_2| = \mu_{S_2}, \quad \text{for every } S_2 \in \mathcal{S}_2.$$

We construct the first cycle. Since  $B_1 \neq B_2$ , there is an element  $e_1 \in B_1 - B_2$ . Let  $e_1 \in S_1$ , for some  $S_1 \in \mathcal{S}_1$ . As  $|B_1 \cap S_1| = |B_2 \cap S_1|$ , there must be another element  $e_2 \in S_1$  such that  $e_2 \in B_2 - B_1$ . Now, let  $e_2 \in S_2$  for some  $S_2 \in \mathcal{S}_2$ . By a similar argument, there must be another element  $e_3 \in S_2$  such that  $e_3 \in B_1 - B_2$ . We keep finding such elements, alternatively from  $B_1 - B_2$  and  $B_2 - B_1$ , until we get back to an element already seen. These elements define the first cycle  $C$ .

For the next cycle, we iterate the above procedure, but switch to  $B'_1 = B_1 \Delta C$  instead of  $B_1$ . Note that  $B'_1$  might not be a base anymore. But by the construction of  $C$ , equations (3.8) and (3.9) still hold for  $B'_1$  and  $B_2$ . This suffices for our purpose. The construction halts when  $B'_1 = B_2$ .  $\square$

Note that there can be cycles which do not come from a symmetric difference of two bases. This is just like the fact that in a bipartite graph, there can be cycles which do not come from a symmetric difference of two perfect matchings. Let  $\mathcal{C}_F$  denote the family of all cycles with respect to face  $F$ . By Lemma 3.7, we have  $\mathcal{C}_F \neq \emptyset$ , for any face  $F$  of dimension  $\geq 1$ .

**COROLLARY 3.10.** *If  $\mathcal{C}_F = \emptyset$ , then  $F$  has dimension 0, i.e.,  $F$  is just a point.*

Consider a face  $F' \subseteq F$ . All equations that hold for  $F$  also hold for  $F'$ . Therefore the partitions of  $E$  that we get from  $F'$  will be refinements of those from  $F$ . Hence, when we go to a sub-face, cycles are only destroyed; no new cycles are created.

**LEMMA 3.11.** *Let  $F, F'$  be two faces of  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  such that  $F' \subseteq F$ . Then  $\mathcal{C}_{F'} \subseteq \mathcal{C}_F$ .*

Thus, the strategy is to successively eliminate cycles to reach smaller and smaller faces, until we reach a face  $F$  where  $\mathcal{C}_F = \emptyset$ . For this purpose, we define the *circulation* of a cycle.

**DEFINITION 3.12.** *For a weight assignment  $w: E \rightarrow \mathbb{Z}$ , the circulation  $c_w(C)$  of a cycle  $C = (e_1, e_2, \dots, e_k)$  is defined as the alternating sum*

$$c_w(C) = |w(e_1) - w(e_2) + w(e_3) - \dots - w(e_k)|.$$

Let  $B_1, B_2$  be two common bases with  $w(B_1) = w(B_2)$  such that  $C = B_1 \triangle B_2$  is a cycle. Then, we have  $c_w(C) = |w(B_1) - w(B_2)| = 0$ . Our next lemma generalizes this observation to *all* cycles in a minimum weight face  $F$ . It essentially argues that the alternating vector corresponding to a cycle is parallel to the face, and thus, is orthogonal to the weight vector.

**LEMMA 3.13.** *Let  $F$  be a face of the polytope  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . Let  $w: E \rightarrow \mathbb{Z}$  be a weight function such that  $w \cdot x$  is constant on  $F$ . Then  $c_w(C) = 0$ , for any  $C \in \mathcal{C}_F$ .*

**PROOF.** Let  $C = (e_1, e_2, \dots, e_{2r}) \in \mathcal{C}_F$ . We split  $C$  into two sets,  $C_1 = \{e_1, e_3, \dots, e_{2r-1}\}$  and  $C_2 = \{e_2, e_4, \dots, e_{2r}\}$ . Now, define the *circulation vector*  $\delta_C \in \mathbb{R}^E$  for cycle  $C$  as

$$\delta_C = \chi^{C_1} - \chi^{C_2},$$

where  $\chi^{C_1}, \chi^{C_2} \in \mathbb{R}^E$  are the characteristic vectors for the sets  $C_1$  and  $C_2$ , respectively. Vector  $\delta_C$  has alternating entries  $+1$  and  $-1$  on the cycle elements, and zeros elsewhere. Note that  $c_w(C) = |w \cdot \delta_C|$ . We will show that  $w \cdot \delta_C = 0$ .

Let  $\{a_1, a_2, \dots, a_p\}$  be the set of corners of  $F$ . Consider their average  $a = (a_1 + a_2 + \dots + a_p)/p$ . Clearly,  $a \in F$ . Now we move from point  $a$  along the vector  $\delta_C$  and go to a new point  $b = a + \epsilon \delta_C$ , for some  $\epsilon \in \mathbb{R}$ . We claim that  $b \in F$ , for small enough  $\epsilon > 0$ . If this is true then  $w \cdot a = w \cdot b$ . By the definition of  $b$ , we get

$$w \cdot a = w \cdot (a + \epsilon \delta_C).$$

We conclude that  $w \cdot \delta_C = 0$ , which proves the lemma.

It remains to argue that  $b \in F$ . Consider an inequality which is not tight for  $F$ . Then, it will not be tight for  $a$  too, because  $a$  is the centroid of  $F$ . One can choose  $\epsilon > 0$  to be small enough so that the inequality remains non-tight for  $b$ . So, we only need to care about the tight equalities for  $F$ ,

$$\begin{aligned} S_0 &= \{e \in E \mid x_e = 0 \quad \forall x \in F\}, \\ \mathcal{T}_1 &= \{T \subseteq E \mid x(T) = r_1(T) \quad \forall x \in F\}, \\ \mathcal{T}_2 &= \{T \subseteq E \mid x(T) = r_2(T) \quad \forall x \in F\}. \end{aligned}$$

We will show that  $b$  satisfies all these constraints. Consider an element  $e \in S_0$ . By definition of  $a$ , we have  $a_e = 0$ . We already remarked above, that  $e$  cannot be a part of a cycle. Therefore, we have  $b_e = a_e$ , and hence  $b_e = 0$ .

Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be the two partitions of  $E$  as in [Lemma 3.5](#). From the definition of a cycle we know that  $|C_1 \cap S| = |C_2 \cap S|$ , for any  $S \in \mathcal{S}_1$ . Thus,

$$\delta_C(S) = 0, \text{ for all } S \in \mathcal{S}.$$

Let  $R \in \mathcal{T}_1$ . By [Lemma 3.5](#),  $R$  is the disjoint union of sets from  $\mathcal{S}_1$ . Hence, we conclude that  $\delta_C(R) = 0$ . Therefore,

$$b(R) = a(R) + \epsilon \delta_C(R) = a(R) = r_1(R).$$

This shows that  $b$  satisfies the second set of constraints. Similarly, one can show that  $b$  satisfies the third set of constraints.  $\square$

Let  $C$  be a cycle, say in  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ , and let  $w$  be a weight function such that  $c_w(C) \neq 0$ . Let  $F$  be the face we get by minimizing  $w$  over  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . It follows that  $w \cdot x$  is constant on  $F$ . Hence from [Lemma 3.13](#), we get that  $C \notin \mathcal{C}_F$ . This means that if  $w$  ensures nonzero circulation for *all* cycles in  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ , then all cycles will be eliminated, i.e.,  $\mathcal{C}_F = \emptyset$  and  $F$  will be a corner. Thus,  $w$  would be isolating. However, we cannot achieve nonzero circulation for all cycles at once, as there are exponentially many possible cycles.

We get around this problem by constructing the weight function in rounds. In each round, we double the length of the eliminated cycles and reach a face of smaller dimension. Thus, in  $\log m$  rounds, we eliminate all cycles and reach a corner. The following lemma shows that the number of cycles we handle in each round remains small. A similar lemma for the number of cycles in a graph was proved by [Fenner et al. \(2016\)](#). Our proof follows a similar line of argument.

**LEMMA 3.14.** *Let  $F$  be a face of  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . If  $\mathcal{C}_F$  has no cycles of length  $\leq r$ , for some even number  $r \geq 2$ , then  $\mathcal{C}_F$  has  $\leq m^4$  cycles of length  $\leq 2r$ .*

PROOF. Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be the two partitions of  $E$  as in [Lemma 3.5](#). Let  $C = (e_0, e_1, \dots, e_{s-1})$  be a cycle of length  $s \leq 2r$ . We choose four elements from the cycle  $C$  which divide it into four almost equal parts: Let  $(a, b, c, d) = (0, \lceil s/4 \rceil, \lceil 2s/4 \rceil, \lceil 3s/4 \rceil)$ . We associate the tuple  $(e_a, e_b, e_c, e_d)$  with cycle  $C$ . Since we could choose cycle  $C$  with any of its element as a starting point, the ordered tuple associated with  $C$  is not uniquely defined. However, we claim that the tuple uniquely describes  $C$ .

CLAIM 3.15. *Cycle  $C$  is the only cycle in  $\mathcal{C}_F$  of length  $\leq 2r$  that is associated with  $(e_a, e_b, e_c, e_d)$ .*

PROOF. Suppose  $C' = (f_0, f_1, \dots, f_{t-1})$  is another such cycle of length  $t \leq 2r$ . We will show that there exists a cycle of length  $\leq r$ , which will be a contradiction.

Let  $(a', b', c', d') = (0, \lceil t/4 \rceil, \lceil 2t/4 \rceil, \lceil 3t/4 \rceil)$ . From the assumption,  $e_0 = f_0$ ,  $e_b = f_{b'}$ ,  $e_c = f_{c'}$  and  $e_d = f_{d'}$ . Without loss of generality, let  $C$  and  $C'$  differ in their first segment. Let  $0 < p < b, b'$  be the first index such that  $e_p \neq f_p$ . Let  $p < q \leq b$  be the first index such that  $e_q = f_h$  for some  $p < h \leq b'$ . As  $e_{p-1} = f_{p-1}$ ,  $e_p$  and  $f_p$  both belong to some common  $S_1 \in \mathcal{S}_1$  or  $S_2 \in \mathcal{S}_2$ .

We consider two cases:

- (i)  $q$  and  $h$  have the same parity: because  $e_q = f_h$ ,  $e_{q-1}$  and  $f_{h-1}$  belong to some common  $S$  or  $T$ . Hence,  
 $(e_p, e_{p+1}, \dots, e_{q-1}, f_{h-1}, f_{h-2}, \dots, f_p)$  forms a valid cycle.
- (ii)  $q$  and  $h$  have a different parity: then the sequence  
 $(e_p, e_{p+1}, \dots, e_{q-1}, f_h, f_{h-1}, \dots, f_p)$  forms a valid cycle as  $e_{q-1}$  and  $f_h$  both belong to some common  $S$  or  $T$ .

The cycles we get in both cases have length  $\leq q - p + h - p + 1 \leq b - 1 + b' \leq r$ . This proves the claim.  $\square$

There are at most  $m^4$  ways to choose the tuple  $(e_a, e_b, e_c, e_d)$ . By [Claim 3.15](#), this gives a bound on the number of cycles of length  $\leq 2r$ .  $\square$

There are standard techniques to give nonzero weights to a small number of sets (see, for example, [Fredman et al. \(1984\)](#)).

**LEMMA 3.16.** *For any number  $s$ , one can construct a set of  $O(m^2s)$  integer weight functions on the set  $[m]$  with weights bounded by  $O(m^2s)$  in NC such that for any set of  $s$  cycles, one of the weight functions will give nonzero circulation to each of the  $s$  cycles.*

For a proof, see [Fenner et al. \(2019, Lemma 2.3\)](#). We apply [Lemma 3.16](#) to a set of  $s = m^4$  cycles. Then, in each round, we get a set of  $O(m^6)$  weight functions, each bounded by  $O(m^6)$ .

**3.4. Isolating weight construction.** Now, we are ready to describe the construction of the isolating weight assignment. The construction goes on similar lines as that in [Fenner et al. \(2016\)](#). They use successive rounds of cycle eliminations from the graph to reach a unique minimum weight perfect matching. We follow the same strategy with the notion of cycles defined in [Section 3.3](#) and reach a unique minimum weight common base.

Let the two given matroids be  $(E, \mathcal{I}_1)$  and  $(E, \mathcal{I}_2)$  with family of base sets  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , respectively. Let  $m = |E|$  and  $t = \lceil \log m \rceil$ . We will define a sequence of weight functions and faces of  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . Let  $F_0 = P(\mathcal{B}_1 \cap \mathcal{B}_2)$ . For  $i = 0, 1, \dots, t-1$ , define

$w_i$ : a weight assignment such that  $c_{w_i}(C) \neq 0$ , for any cycle  $C \in \mathcal{C}_{F_i}$  of length  $\leq 2^{i+1}$ ,

$F_{i+1}$ : the set of points in  $F_i$  minimizing the weight function  $w_i$ .

We combine the weight functions  $w_0, w_1, \dots, w_{t-1}$  with decreasing precedence. Let  $N$  be a number that is larger than the weight of any point in  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  with respect to any of these weight functions. We will see later that choosing  $N = O(m^7)$  suffices. For  $i = 0, 1, \dots, t-1$ , define

$$W_i = w_0 N^i + w_1 N^{i-1} + \dots + w_i N^0.$$

Our final weight assignment will be  $W_{t-1}$ .

**CLAIM 3.17.**  *$F_{i+1}$  is the set of minimum points in  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  with respect to  $W_i$ , for  $i = 0, 1, \dots, t-1$ .*



PROOF. We prove this by induction. The claim is clearly true for  $i = 0$ . Now, assume that  $F_i$  is the set of points in  $P(\mathcal{B}_1 \cap \mathcal{B}_2)$  that minimizes  $W_{i-1}$ . Then  $F_i$  is also the set of points that minimizes  $NW_{i-1}$ . As  $NW_{i-1}$  always dominates  $w_i$ , the set of points that minimizes  $W_i = NW_{i-1} + w_i$  will be a subset of  $F_i$ . This subset is exactly those points in  $F_i$  where  $w_i$  is minimized, that is  $F_{i+1}$ .  $\square$

CLAIM 3.18.  $\mathcal{C}_{F_i}$  has no cycles of length  $\leq 2^i$ , for  $i = 1, 2, \dots, t$ .

PROOF. By the definition of  $w_{i-1}$ ,  $c_{w_{i-1}}(C) \neq 0$  for any cycle  $C \in \mathcal{C}_{F_{i-1}}$  of length  $\leq 2^i$ . As  $w_{i-1}$  is constant over the face  $F_i$ , we have  $c_{w_{i-1}}(C) = 0$ , for all cycles  $C \in \mathcal{C}_{F_i}$ , by Lemma 3.13. Recall Lemma 3.11 that  $\mathcal{C}_{F_i} \subseteq \mathcal{C}_{F_{i-1}}$ . Thus,  $\mathcal{C}_{F_i}$  has no cycles of length  $\leq 2^i$ .  $\square$

LEMMA 3.19. Weight function  $W_{t-1}$  is isolating.

PROOF. By Claim 3.17, the face minimized by  $W_{t-1}$  is  $F_t$ . By Claim 3.18,  $\mathcal{C}_{F_t}$  has no cycles of length  $\leq 2^t = m$ . That is,  $\mathcal{C}_{F_t} = \emptyset$ . By Corollary 3.10,  $F_t$  has only one corner, i.e.,  $W_{t-1}$  is isolating.  $\square$

Since  $\mathcal{C}_{F_i}$  has no cycles of length  $\leq 2^i$  (Claim 3.18), the number of cycles in  $\mathcal{C}_{F_i}$  of length  $\leq 2^{i+1}$  is at most  $m^4$  (Lemma 3.14). Thus,  $w_i$  needs to give nonzero circulations to at most  $m^4$  cycles. By Lemma 3.16, each  $w_i$  has weights bounded by  $O(m^6)$ . Hence, it is sufficient to choose  $N$  to be  $O(m^7)$ . It follows that  $W_{t-1}$  will have weights bounded by  $O(m^{7 \log m})$ . By Lemma 3.16, we get  $O(m^6)$  possible weight functions for each  $w_i$ , and therefore  $O(m^{6 \log m})$  combinations for  $W_{t-1}$ . We need to try all of them in parallel.

LEMMA 3.20. For a given number  $m$ , we can construct  $O(m^{6 \log m})$  weight functions on  $[m]$  with weights bounded by  $O(m^{7 \log m})$  such that for any matroid intersection on the ground set  $[m]$ , one of the weight functions isolates a common base.

As mentioned in [Section 2.4](#), by plugging-in a isolating weight assignment in the determinant polynomial we can decide whether there exists a common base, and construct it if one exists. We will try all quasi-polynomially many weight assignments given by [Lemma 3.20](#) in parallel, and we know one of them will be isolating. As our weights are quasi-polynomially bounded, the entries in the matrix  $D(z)$  from [Section 2.4](#) will be univariate polynomials with quasi-polynomially large degree. Thus, the determinant can be computed in quasi-NC ([Berkowitz 1984](#); [Borodin et al. 1984](#)). This proves [Theorem 3.1](#).

**Proof of Remark 3.2: Getting precise bounds of quasi-NC<sup>4</sup> in general and quasi-NC<sup>2</sup> for zero characteristic.** The determinants of matrices that have polynomially large entries (possibly from the polynomial ring) can be computed in NC<sup>2</sup> ([Berkowitz 1984](#); [Borodin et al. 1984](#)). Applying this naively to our setting where the entries are as large as  $m^{O(\log m)}$ , we get a bound of quasi-NC<sup>4</sup> for computing the determinant. We can further reduce this bound to quasi-NC<sup>2</sup> in the case of fields of characteristic zero using the Chinese Remainder Theorem.

Let the two matrices  $U$  and  $V$  representing the two matroids have their entries bounded by  $\text{poly}(m)$  in bit-size. It follows that  $\det(D)(z)$  will be a polynomial of degree  $m^{O(\log m)}$  and its coefficients will have bit-size at most  $\text{poly}(m)$ . Let  $2^M$  be the least power of two larger than the absolute values of these coefficients. Replace  $z$  with  $2^M$ . Then  $\det(D)(z = 2^M)$  will remain nonzero. Moreover, the highest power of  $2^M$  that divides  $\det(D)(z = 2^M)$  will be  $2^{Mw^*}$ , where  $w^*$  is the weight of the unique minimum weight common base.

The problem in computing  $\det(D)(z = 2^M)$  is that the matrix entries will have quasi-polynomial size. This issue can be solved by first computing  $\det(D)(z = 2^M)$  modulo small prime numbers and then finding  $\det(D)(z = 2^M)$  using Chinese remaindering. This enables us to compute the determinant in quasi-NC<sup>2</sup> (see [Fenner et al. \(2019, Section 3\)](#) for more details). Once we have  $\det(D)(z = 2^M)$ , we can find out  $w^*$  and use it to construct the minimum weight common base as described in [Section 2.4](#). Then, the overall bound

stays quasi-NC<sup>2</sup>.

## 4. Applications

We already mentioned the connection of our isolating weight construction to *Polynomial Identity Testing* in [Section 2.4](#). In this section, we extend the class of polynomials even further where our technique applies. Then, we show that this extended class of polynomials can be used to solve the *matroid union problem* in quasi-NC.

**4.1. Polynomial identity testing (PIT).** The weight assignment constructed in [Lemma 3.20](#) yields a quasi-polynomial time blackbox identity test, i.e., a hitting set, for polynomials of the form  $D = UZV^\top$ , where  $U, V$  are  $n \times m$  matrices over some field  $\mathbb{F}$ , and  $Z$  is a  $m \times m$  diagonal matrix with  $Z_{i,i} = z_i$ , for  $i = 1, 2, \dots, m$ . To see this, recall from [Section 2.4](#) that if  $w$  is isolating for the common bases of  $U$  and  $V$ , then the univariate polynomial  $\det(D)(z)$ , obtained after substituting  $z_e = z^{w(e)}$ , for each  $e \in [m]$ , is nonzero. Since  $w$  has weights bounded by  $m^{O(\log m)}$ , the degree of the polynomial  $\det(D)(z)$  is bounded by  $m^{O(\log m)}$ . Thus, any set of  $m^{O(\log m)}$  field elements constitutes a hitting set for  $\det(D)(z)$ .

Let  $u_i$  and  $v_i$  be the  $i$ -th columns of  $U$  and  $V$ , respectively. Then, we can rewrite  $D$  as  $D = \sum_{i=1}^m z_i u_i v_i^\top$ . Note that any rank-1 matrix is of the form  $uv^\top$  for some  $u, v \in \mathbb{F}^n$ . Thus, we get the following corollary.

**COROLLARY 4.1.** *In quasi-polynomial time, one can compute a hitting set of quasi-polynomial size for polynomials of the form  $\det(\sum_{i=1}^m z_i A_i)$ , where  $A_i$  is an  $n \times n$  matrix of rank at most 1 over some field  $\mathbb{F}$ , for  $i = 1, 2, \dots, m$ .*

We can further generalize the class of polynomials we can handle and add an arbitrary constant matrix  $A_0$ , i.e., with no rank restriction.

**THEOREM 4.2.** *There is an  $(m+n)^{O(\log(m+n))}$ -size hitting set (computable in quasi-polynomial time) for polynomials of form  $\det(A_0 +$*

$\sum_{i=1}^m z_i A_i$ ), where  $A_i$  is an  $n \times n$  matrix over some field  $\mathbb{F}$ , for  $i \geq 0$ , and is of rank at most 1, for  $i \geq 1$ .

Let  $U$  and  $V$  be the matrices from above such that

$$A_0 + \sum_{i=1}^m z_i A_i = A_0 + UZV^\top.$$

Observe that the entries of this matrix are linear forms in the variables  $z_1, z_2, \dots, z_m$ . The following lemma constructs a matrix  $M$  such that  $\det(A_0 + UZV^\top) = \det(M)$  and the entries of  $M$  are either constant or a single variable  $z_i$ . Moreover, every variable  $z_i$  occurs only once in  $M$ . This *rank-one to read-once* reduction is due to Matthew Anderson, Amir Shpilka and Ben Lee Volk.

LEMMA 4.3 ([Anderson et al. 2016](#)).

$$(4.4) \quad \det(A_0 + UZV^\top) = \det \begin{pmatrix} I & Z & 0 \\ 0 & I & V^\top \\ U & 0 & A_0 \end{pmatrix}.$$

PROOF. Let  $A, B, C, D$  be matrices where  $A$  and  $D$  are square matrices and  $A$  is invertible. Then, we have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}$$

and hence,

$$(4.5) \quad \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B).$$

We split the matrix on the right-hand side of (4.4) into

$$A = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ V^\top \end{pmatrix}, \quad C = (U \ 0), \quad D = A_0$$

and apply Equation (4.5). We have  $\det(A) = 1$ . Note that  $A^{-1} = \begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix}$ , and therefore we get  $D - CA^{-1}B = A_0 + UZV^\top$ . This proves the lemma.  $\square$

Murota (1993) has shown that PIT for read-once matrices reduces to the matroid intersection problem. We present the reduction in a way that is suitable for blackbox identity testing. Let  $Q(\mathbf{z}) = \det(A_0 + UZV^\top)$ . By Lemma 4.3, polynomial  $Q(\mathbf{z})$  is multilinear.

The first step is to *homogenize*  $Q(\mathbf{z})$ . Consider the polynomial

$$\begin{aligned} Q'(z_1, z_2, \dots, z_{2m}) \\ = z_{m+1}z_{m+2} \cdots z_{2m} \cdot Q(z_1/z_{m+1}, z_2/z_{m+2}, \dots, z_m/z_{2m}), \end{aligned}$$

where  $z_{m+1}, z_{m+2}, \dots, z_{2m}$  are new variables. Observe that  $Q'$  is homogeneous, every monomial in  $Q'$  has degree  $m$ . Note also that  $Q' \neq 0$  if and only if  $Q \neq 0$ . Moreover, if  $Q'$  is nonzero at a point  $(\alpha_1, \alpha_2, \dots, \alpha_{2m})$ , where  $\alpha_{m+1}, \dots, \alpha_{2m} \neq 0$ , then  $Q$  is nonzero at the point  $(\alpha_1/\alpha_{m+1}, \alpha_2/\alpha_{m+2}, \dots, \alpha_m/\alpha_{2m})$ . Thus, it suffices to find a hitting set for  $Q'$ .

Let  $Z'$  be the  $m \times m$  diagonal matrix with  $Z'_{i,i} = z_{m+i}$ . Then, we can write

$$Q'(\mathbf{z}) = \det \begin{pmatrix} Z' & Z & 0 \\ 0 & I & V^\top \\ U & 0 & A_0 \end{pmatrix},$$

Compared with the representation of  $Q$  in (4.4), the matrix here has  $Z'$  in the left upper corner instead of  $I$ . That is, there are only variable entries in the first  $m$  rows, and zeros, but no other constants. We will take advantage of this representation.

Define matrices

$$Y = \begin{pmatrix} 0 & I & V^\top \\ U & 0 & A_0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} Z' & Z & 0 \\ & Y & \end{pmatrix}.$$

Hence  $Q'(\mathbf{z}) = \det(L)$ . Let  $Y_i$  be the  $i$ -th column of  $Y$ , for  $1 \leq i \leq 2m + n$ . Since variables  $z_i$  and  $z_{m+i}$  are in the same row of  $L$ , exactly one of them will appear in any monomial of  $Q'(\mathbf{z})$ , for each  $1 \leq i \leq m$ . For any such monomial  $\prod_{i \in S} z_i$  with  $S \subseteq [2m]$ , its coefficient is nonzero if and only if the columns  $\{Y_i\}_{i \in [2m+n]-S}$  are linearly independent. With these observations, we can show that the monomials of  $Q'(\mathbf{z})$  exactly correspond to the common bases of two matroids: Let  $E = [2m + n]$ .

- The first matroid  $M_1 = (E, \mathcal{I}_1)$  is defined by the  $m \times (2m+n)$  matrix  $\begin{pmatrix} I & I & 0 \end{pmatrix}$ . The matrix has two ones in every row, at position  $i$  and  $i+m$ . Therefore any base set of matroid  $M_1$  has exactly one of the two elements  $i, m+i$ , for each  $1 \leq i \leq m$ , and no elements  $> 2m$ . Let the collection of all its base sets be  $\mathcal{B}_1$ .
- Let matroid  $M_2 = (E, \mathcal{I}_2)$  be defined by the  $(m+n) \times (2m+n)$  matrix  $Y$ . We choose our second matroid to be the *dual matroid*  $M_2^* = (E, \mathcal{I}_2^*)$ . This is because a set  $S \subseteq E$  is a base set of  $M_2^*$  if and only if the set  $\bar{S} = [2m+n] - S$  is a base set in  $M_2$ , that is, the columns  $\{Y_i\}_{i \in [2m+n]-S}$  form a basis for  $Y$ . Let the collection of all base sets of  $M_2^*$  be  $\mathcal{B}_2^*$ .

Now the monomials in  $Q'(\mathbf{z})$  exactly correspond to the sets in  $\mathcal{B}_1 \cap \mathcal{B}_2^*$ . Thus, we can construct an isolating weight assignment for the monomials of  $Q'(\mathbf{z})$ , which gives us a hitting set. As we have to try quasi-polynomially many weight assignments, our hitting set size is quasi-polynomial. This proves [Theorem 4.2](#).

**4.2. Maximum rank matrix completion.** In the maximum rank matrix completion problem, we are given an  $n \times n$  partially filled matrix and the goal is to complete the matrix so as to get the maximum possible rank. The problem reduces to PIT as follows. Consider the matrix  $A$  obtained by filling in a distinct variable  $z_i$  for each of the  $m \leq n^2$  blank entries in the given matrix. Note that  $A$  can be written in the form  $A_0 + \sum_{i=1}^m z_i A_i$ , where  $A_0$  is the matrix with the already filled in entries, and 0 at all the blank positions. Matrix  $A_i$  has entry 1 at the  $i$ -th blank position and 0 at all other entries, for  $1 \leq i \leq m$ . Note that  $A_1, A_2, \dots, A_m$  have rank at most 1. We need to find a substitution for the variables which maximizes the rank of  $A$ .

Let  $\mathbf{z} = (z_1, z_2, \dots, z_m)$  and  $\mathbf{z} = \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{F}^m$  be a substitution that achieves the maximum rank, say  $r$ . Then, there is an  $r \times r$  sub-matrix  $B$  of  $A$  such that  $\det(B(\mathbf{z} = \boldsymbol{\alpha}))$  is nonzero. Thus, the polynomial  $\det(B(\mathbf{z}))$  must be nonzero as well. Note that the matrix  $B$  has the same form as  $A$ , that is,  $B = B_0 + \sum_{i=1}^m z_i B_i$ , where  $B_1, B_2, \dots, B_m$  are rank  $\leq 1$  matrices.

Therefore, our hitting set constructed in [Theorem 4.2](#) works for  $\det(B)$ . That is, when the variables  $\mathbf{z}$  are substituted with points  $\boldsymbol{\alpha}$  from our hitting set,  $\det(B(\mathbf{z} = \boldsymbol{\alpha}))$  will be nonzero for at least one of the points  $\boldsymbol{\alpha}$ . For such a point  $\boldsymbol{\alpha}$ , matrix  $A(\mathbf{z} = \boldsymbol{\alpha})$  has the maximum rank.

Note however that we do not know which point in the hitting set will achieve the maximum rank. It turns out that we can combine all the points in the hitting set to construct one fixed substitution which achieves the maximum rank for all possible choices of the already filled in entries, i.e.,  $A_0$ . For this purpose, one can use the well-known technique of Lagrange interpolation (see, for example, [Forbes \(2014, Lemma 3.2.22\)](#)).

**LEMMA 4.6** (Lagrange Interpolation). *Let  $\mathcal{H} \subset \mathbb{F}^r$  be a hitting set for a family of polynomials  $\mathcal{P} \subseteq \mathbb{F}[z_1, z_2, \dots, z_r]$ . Let  $\{\alpha_h\}_{h \in \mathcal{H}}$  be distinct field elements and  $t$  be a variable. Then, each polynomial  $p$  in  $\mathcal{P}$  has a nonzero evaluation over the field  $\mathbb{F}(t)$  at*

$$L(t) = \sum_{h \in \mathcal{H}} h \frac{\prod_{h' \in \mathcal{H} - \{h\}} (t - \alpha_{h'})}{\prod_{h' \in \mathcal{H} - \{h\}} (\alpha_h - \alpha_{h'})}.$$

**PROOF.** Let  $p \in \mathcal{P}$ . Since  $\mathcal{H}$  is a hitting set, there exists an  $h \in \mathcal{H}$  such that  $p(h) \neq 0$ . Note that  $L(\alpha_h) = h$ . Thus,  $p(L(\alpha_h)) \neq 0$  and hence  $p(L(t)) \neq 0$ .  $\square$

We apply [Lemma 4.6](#) with  $r = m$ . After substituting  $\mathbf{z} = L(t)$  in  $A$ , the entries in the resulting matrix  $A(t)$  are univariate polynomials in  $t$  of quasi-polynomial degree, since our hitting set has quasi-polynomial size. The polynomial  $\det(B(\mathbf{z} = L(t)))$  will be nonzero as a univariate polynomial.

If the underlying field is a characteristic zero field, then we can replace  $t$  with a large enough number  $\gamma$  and ensure that  $\det(B(\mathbf{z} = L(\gamma)))$  remains nonzero. To see this, suppose each entry of the matrix  $A$  has bit-size bounded by  $n$ . Then, the coefficients of the univariate polynomial  $\det(B(\mathbf{z} = L(t)))$  will have bit-sizes bounded by  $\text{poly}(n)$ . If we choose  $\gamma$  to be larger than the absolute value of each of these coefficients, then  $\det(B(\mathbf{z} = L(\gamma)))$  remains nonzero. Thus, it suffices to have  $\gamma$  with bit-size  $\text{poly}(n)$ . After replacing

$t$  with  $\gamma$ , the entries of  $A$  have bit-size  $n^{O(\log n)}$ , since the entries of  $A(\mathbf{z} = L(t))$  are univariate polynomials of degree  $n^{O(\log n)}$ .

Similarly, in case of a finite characteristic field,  $t$  can be replaced by an element from a field extension that has a sufficiently large degree.

**THEOREM 4.7.** *Given a partially filled matrix of size  $n \times n$  and with entries having bit-size  $n$ , there is a quasi-NC algorithm to find a fixed substitution (of bit-size  $n^{O(\log n)}$ ) for the blank entries which maximizes the rank of the matrix, irrespective of where and what the given entries in the matrix are.*

**4.3. Matroid union.** For given matroids  $M_1 = (E_1, \mathcal{I}_1)$ ,  $M_2 = (E_2, \mathcal{I}_2)$ ,  $\dots$ ,  $M_k = (E_k, \mathcal{I}_k)$ , their union  $M = M_1 \vee M_2 \vee \dots \vee M_k$  is defined as  $(E, \mathcal{I})$ , where  $E = \bigcup_{i=1}^k E_i$  and

$$\mathcal{I} = \left\{ \bigcup_{i=1}^k I_i \mid I_i \in \mathcal{I}_i, \text{ for } 1 \leq i \leq k \right\}.$$

It is known that  $M$  is again a matroid, and  $M$  is linear when  $M_1, M_2, \dots, M_k$  are linear; see [Schrijver \(2003\)](#).

The *matroid union problem* is to compute a base of the matroid union  $M$  i.e., to compute independent sets  $I_i \in \mathcal{I}_i$  for  $1 \leq i \leq k$  which maximize  $|I_1 \cup I_2 \cup \dots \cup I_k|$ . The simple greedy algorithm for computing a base of a matroid requires an independence oracle, but here it is not immediately clear how to test if a set is independent in  $M$ . The problem thereby is that the ground sets  $E_i$  may overlap each other. In case that the sets  $E_i$  are pairwise disjoint, one can simply put the matrices  $A_i$  representing  $M_i$  as blocks in a block-diagonal matrix  $A$ . Then  $A$  is a linear representation of  $M$  and we can easily solve the matroid union problem.

In general, the ground sets  $E_i$  overlap each other. Inspired by the disjoint case, the idea now is to work with a *disjoint union* of the ground sets as one matroid and then define a second matroid that allows to take only one copy of an element in the disjoint union. The intersection of the two matroids describes  $M$ . In more detail, the reduction from union to intersection is as follows. We define two matroids  $M'$  and  $M''$ .



1. Matroid  $M' = (E', \mathcal{I}')$  is defined by the *disjoint union* of  $M_1, M_2, \dots, M_k$ . That is,

$$E' = E_1 \sqcup E_2 \sqcup \dots \sqcup E_k = \{ (e, i) \mid e \in E_i \}.$$

A set  $I'$  is independent in  $M'$  if  $I' = I_1 \sqcup I_2 \sqcup \dots \sqcup I_k$ , for  $k$  independent sets  $I_1, I_2, \dots, I_k$  of  $M_1, M_2, \dots, M_k$ , respectively.

2. Matroid  $M'' = (E', \mathcal{I}'')$  is a partition matroid on the same ground set  $E'$ . The sets that partition  $E'$  are the copies of  $e$  in  $E'$ . That is, for  $e \in E$ , define  $B_e = \{ (e, i) \mid e \in E_i \} \subseteq E'$ . A set  $I''$  is independent in  $M''$  if  $|I'' \cap B_e| \leq 1$ , for all  $e \in E$ .

Now observe that for a common independent set  $\widehat{I}$  of  $M'$  and  $M''$ , the projection

$$I = \{ e \in E \mid (e, i) \in \widehat{I} \text{ for some } i \in [k] \}$$

is independent in  $M$ . Conversely, every independent set  $I$  of  $M$  corresponds to a common independent set  $\widehat{I}$  of  $M'$  and  $M''$ . Note that  $|I| = |\widehat{I}|$  in both directions. Note also that  $M'$  and  $M''$  are linear when  $M_1, M_2, \dots, M_k$  are linear.

Hence, the matroid union problem reduces to matroid intersection, and thus has a polynomial-time algorithm (Edmonds 1968; Schrijver 2003). Also, our quasi-NC algorithm for matroid intersection implies a quasi-NC algorithm for matroid union.

**THEOREM 4.8.** *Linear Matroid Union is in quasi-NC.*

In case of linear matroids, another interesting question is to compute a linear representation for the matroid union. Narayanan, Saran & Vazirani (1994) gave a randomized NC-algorithm for computing such a linear representation. It turns out that we can derandomize their algorithm with our PIT result.

The construction of the linear representation is as follows. Suppose the matroids  $M_1, M_2, \dots, M_k$  are given by the matrices  $U_1, U_2, \dots, U_k$ , respectively. Without loss of generality, one can assume that all the matroids have the same ground set, i.e., each  $U_i$  and  $U_j$  have a one-to-one correspondence between their columns. If not,

then one can add extra zero columns to the matrices. We want to find a representation of  $M = M_1 \vee M_2 \vee \cdots \vee M_k$ .

Let the dimensions of  $U_i$  be  $n_i \times m$ , for  $1 \leq i \leq k$ . For each  $1 \leq i \leq k$ , define  $U'_i$  to be an  $n_i \times m$  matrix such that its  $j$ -th column is the  $j$ -th column of  $U_i$ , multiplied by a variable  $z_{i,j}$ . Define the  $(n_1 + n_2 + \cdots + n_k) \times m$  matrix  $V$  by stacking the matrices  $U'_1, U'_2, \dots, U'_k$  one below another,

$$V = \begin{pmatrix} U'_1 \\ U'_2 \\ \vdots \\ U'_k \end{pmatrix}$$

Then, a set  $I$  is independent in  $M$  if and only if the corresponding columns  $V_I$  in  $V$  are linearly independent (over the field  $\mathbb{F}(z_{i,j})_{i,j}$ ) (Narayanan *et al.* 1994, Lemma 3.1). To get a matrix over the base field, one can plug-in random values for the variables  $z_{i,j}$ . This works because a random substitution preserves the nonzeroness of minors with high probability (Demillo & Lipton 1978; Schwartz 1980; Zippel 1979).

Note that in the matrix  $V$ , any variable  $z_{i,j}$  appears only in the  $j$ -th column. Thus, any minor of  $V$  will be a polynomial of the form  $\det(\sum_{i,j} A_{i,j} z_{i,j})$ , where matrix  $A_{i,j}$  has rank 1, for  $1 \leq i \leq k$  and  $1 \leq j \leq m$ . This is precisely the form for which we have given a hitting set in Theorem 4.2. Thus, any nonzero minor of  $V$  will have a nonzero evaluation at least at one point of the hitting set.

However, this does not yet solve our problem because we need to find one substitution which works simultaneously for *all* nonzero minors. For this, one can again use the technique of Lagrange interpolation, as in the previous subsection (Lemma 4.6).

We apply Lemma 4.6 (with  $r = km$ ) to combine the hitting-set into one substitution. After substituting the variables  $(z_{i,j})_{i,j}$  in  $V$  by  $L(t)$  (from Lemma 4.6), the entries in the resulting matrix  $V(t)$  are univariate polynomials in  $t$  of quasi-polynomial degree, since our hitting set has quasi-polynomial size. As argued in the previous subsection,  $t$  can be replaced with a field value of size  $\text{poly}(m)$  (or of large enough degree in case of finite characteristic) while preserving the nonzeroness of each minor, where the matrices  $M_1, M_2, \dots, M_k$

have their entry sizes bounded by  $\text{poly}(m)$ . Finally, after substituting  $t$  with such a large enough value, the entries in  $V$  have quasi-polynomial size since the univariate polynomials in  $V(t)$  have quasi-polynomial degree.

**THEOREM 4.9.** *Given linear matroids  $M_1, M_2, \dots, M_k$  each with ground set size  $m$ , there is a quasi-NC algorithm to compute a linear representation  $V$  of  $M = M_1 \vee M_2 \vee \dots \vee M_k$ , where the entries of matrix  $V$  are of size  $2^{O(\log^2(mk))}$ .*

**4.4. Some more combinatorial problems.** To illustrate the wide range of matroid intersection, we give a few more examples of combinatorial problems which are known to reduce to linear matroid intersection (see [Schrijver \(2003\)](#)).

**Two edge-disjoint spanning trees in a graph.** This problem can be reduced to the intersection of a graphic matroid and a cographic matroid. Recall that for an undirected connected graph  $G = (V, E)$ , the graphic matroid has ground set  $E$  and any forest in  $G$  is an independent set. Thus, any spanning tree in  $G$  is a base set. In the cographic matroid of  $G$ , a set of edges is independent if its removal keeps the graph connected. Both the matroids are known to be linear (see, e.g., [Oxley \(2006\)](#)). Now, to find two edge-disjoint spanning trees in  $G$ , we find the maximum edge set which is independent in both, the graphic and the cographic matroid. This will be a spanning tree whose removal keeps the graph connected. Thus, after removing this tree, we can find another spanning tree in the resulting graph.

**Rainbow spanning tree in an edge-colored graph.** Given a graph with colored edges, the problem asks if there is a spanning tree with all its edges having distinct colors. To capture this by matroid intersection, define the first matroid to be the graphic matroid of  $G$ . The second matroid is a partition matroid, where each set of the partition consists of the edges of one color. A set is independent if it contains at most one edge of a color.

**Arborescence in a directed graph.** An *arborescence* is a directed acyclic graph, that has a vertex  $r$ , called the *root*, such that for any vertex  $v$ , there is exactly one path from  $r$  to  $v$ . In the  $r$ -arborescence problem we have given a directed graph  $G$  and a vertex  $r$ . The task is to find an arborescence in  $G$  with root  $r$ .

Note that an  $r$ -arborescence is any set of edges which form a spanning tree in the underlying undirected graph and for each vertex  $v$  other than the root, it has exactly one edge incoming to  $v$ . Hence, we can express the problem as the intersection of two matroids: The first matroid is the graphic matroid of the underlying undirected graph. The second matroid is a partition matroid with the partition  $\cup_v E_v$ , where  $E_v$  is the sets of edges incoming to the vertex  $v$ . (We may assume that there are no edges incoming to the root  $r$ ).

**Shortest  $R$ - $S$  biconnector and a longest  $R$ - $S$  biferest of a graph.** For a graph  $G = (V, E)$  and a partition of  $V$  into  $R$  and  $S$ , an  $R$ - $S$  *biconnector* is a set  $F \subseteq E$ , such that each component of  $(V, F)$  intersects both  $R$  and  $S$  (Schrijver 2003, Chapter 54). If  $F$  has minimum size, it is a *shortest biconnector*.

A *spanning set* in a matroid is a set which contains a base set. In case of graphic matroids, a spanning set is any subset of edges which forms a connected graph. For two matroids on the same ground set, a *common spanning set* is a subset of the ground set that is a spanning set in both the matroids. Finding a shortest  $R$ - $S$  biconnector reduces to finding a shortest common spanning set of two matroids.

The reduction goes as follows. Define graph  $G_R$  from  $G$  by contracting the set  $R$  to one vertex. The edges within  $R$  are kept as self-loops. Define graph  $G_S$  similarly for  $S$ . Let  $M_R$  and  $M_S$  be the graphic matroid for  $G_R$  and  $G_S$ , respectively. Now observe that an edge set in  $G$  is a  $R$ - $S$  biconnector if and only if it is a spanning set in both  $M_R$  and  $M_S$ .

To reduce the problem of shortest common spanning set to maximum common independent set, recall that the complement of a spanning set is an independent set of the dual matroid. Thus, the problem reduces to maximum common independent set of the

two dual matroids.

An  $R$ - $S$  *biforest* is a forest  $F$  such that each component of  $(V, F)$  has at most one edge in the cut  $\delta(R)$ . The reduction  $R$ - $S$  biforest to linear matroid intersection is similar as above. Let  $u$  be a vertex in  $R$  and  $v$  be a vertex in  $S$ . Define the graph  $G_u$  from  $G$  as follows: for any edge  $e$  which has one endpoint in  $R$  and the other in  $S$ , change the endpoint in  $R$  to  $u$ . Define graph  $G_v$  similarly with respect to  $S$  and  $v$ . Then, a common independent set of the two graphic matroids corresponding to  $G_u$  and  $G_v$  is precisely an  $R$ - $S$  biforest.

## 5. Discussion

One of the main open questions is to do isolation with polynomially bounded weights, or to come up with a different NC-algorithm for linear matroid intersection. It would be interesting to find out for what polytopes our isolation technique works. For general matroids, the parallel complexity of matroid intersection is not clear. Can we find an NC algorithm (randomized or deterministic) for the general case.

A generalization of matroids are *polymatroids*. These are polytopes similar to the matroid polytope, where instead of the rank function one can use any submodular function that is nonnegative and nondecreasing. The key argument in our construction is the structure of the faces of the matroid intersection polytope, which basically comes from [Lemma 3.3](#). Note that for the proof of this lemma, the only property used was submodularity of the rank function. Thus, one can verify that the whole argument generalizes to polymatroid intersection. That is, our weight function isolates a corner in a polymatroid intersection polytope.

Another generalization of matroid intersection is matroid matching, which also captures perfect matchings in general graphs (not necessarily bipartite). It is an open question to find a quasi-NC algorithm for linear matroid matching.

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